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On the numerical evaluation of an oscillating infinite series

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Abstract: An oscillating infinite series involving products of the Bessel function $J_0(x)$ is evaluated. The series is transformed first into the sum of three infinite integrals by using contour integration and then the infinite integral with oscillating integrand is transformed by some identities into an expression containing modified Bessel functions only. Finally, this expression and the other infinite integrals are evaluated numerically without any computational difficulties at all.

Keywords: Oscillating infinite series, transformations.

1. Introduction

This paper is concerned with the evaluation (and tabulation) of infinite series containing products of the Bessel function $J_0(x)$ (first kind Bessel function of order zero) of the form

$$A = \sum_{m=1,3}^{\infty} m\left(\frac{m}{\sqrt{m^2+u^2}} - 1\right) J_0(am) J_0(bm), \tag{1}$$

where u, a and b are constants.

This series is transformed first into the sum of three infinite integrals by using contour integration; then, of these three integrals the one which contains an oscillating term with products of the Bessel function $J_0(x)$ of the form

$$B = \int_0^{\infty} t\left(\frac{t}{\sqrt{t^2+u^2}} - 1\right) J_0(at) J_0(bt)\, \mathrm{d}t \tag{2}$$

is transformed by some identities in [3] into the form of a finite integral containing modified Bessel functions only. So, the difficulties associated with the computation of an oscillating infinite series and/or similarly the computation of an infinite integral with oscillating integrand are avoided. The resulting finite and infinite integrals are evaluated numerically using Gauss' formula for various values of u, a and b and these results are tabulated.

It happens frequently that the solutions of mixed boundary value problems in mathematical physics can be reduced to the solutions of integral equations in which the kernels and source

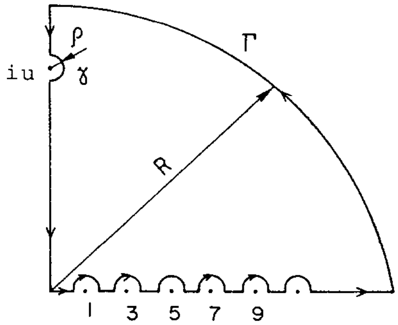

Fig. 1.

functions contain infinite series of the above considered type (e.g., magnetohydrodynamic flow problems in ducts, crack problems in elasticity,...). Because of the oscillating nature of J_0 such infinite series are extremely hard to evaluate on the computer, particularly for large values of u. However, this constant u may be quite large depending on the physical situation of the problem [1,4]. The parameter u usually contains either the Reynolds number in fluid dynamics problems or the Hartmann number in magnetohydrodynamic flow problems and may take values up to 500 in some cases.

2. Transformation of the infinite series

For the transformation of infinite series (1) we consider the function

$$F(z)=z\left(\frac{z}{\sqrt{z^2+u^2}}-1\right)J_0(az)J_0(bz)\exp\left(\frac{\mathrm{i}\pi z}{2}\right)\sec\left(\frac{\pi z}{2}\right),$$

taken along the contour shown in Fig. 1, and taking the branch of the function $(z^2+u^2)^{-1/2}|_{z=0}=u^{-1}$. We can show that

$$\int_\Gamma F(z)\,\mathrm{d}z=0 \quad \text{and} \quad \int_\gamma F(z)\,\mathrm{d}z=0$$

in the limit as $R\to\infty$ with $a+b<\pi$ and $\rho\to 0$ respectively.

Hence we have

$$\int_0^\infty t\left(\frac{t}{\sqrt{t^2+u^2}}-1\right)J_0(at)J_0(bt)\exp\left(\frac{\mathrm{i}\pi t}{2}\right)\sec\left(\frac{\pi t}{2}\right)\mathrm{d}t$$
$$+\lim_{\rho\to 0}\sum_{m=1,3}^{\infty}\int_\pi^0 m\left(\frac{m}{\sqrt{m^2+u^2}}-1\right)J_0(am)J_0(bm)\exp\left(\frac{\mathrm{i}\pi}{2}(m+\rho\,\mathrm{e}^{\mathrm{i}\theta})\right)$$
$$\times\sec\left(\frac{\pi}{2}(m+\rho\,\mathrm{e}^{\mathrm{i}\theta})\right)\mathrm{i}\rho\,\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta$$
$$+\int_\infty^{u_+}\mathrm{i}s\left(\frac{\mathrm{i}s}{\sqrt{u^2-s^2}}-1\right)J_0(\mathrm{i}as)J_0(\mathrm{i}bs)\exp\left(-\frac{\pi s}{2}\right)\operatorname{sech}\left(\frac{\pi s}{2}\right)\mathrm{i}\,\mathrm{d}s$$
$$+\int_{u^-}^0\mathrm{i}s\left(\frac{\mathrm{i}s}{\sqrt{u^2-s^2}}-1\right)J_0(\mathrm{i}as)J_0(\mathrm{i}bs)\exp\left(-\frac{\pi s}{2}\right)\operatorname{sech}\left(\frac{\pi s}{2}\right)\mathrm{i}\,\mathrm{d}s=0.$$

So,

$$
\begin{aligned}
&\int_0^\infty t\left(\frac{t}{\sqrt{t^2+u^2}}-1\right)J_0(at)J_0(bt)\left(1+\mathrm{i}\tan\frac{\pi t}{2}\right)\mathrm{d}t\\
&-2\sum_{m=1,3}^{\infty} m\left(\frac{m}{\sqrt{m^2+u^2}}-1\right)J_0(am)J_0(bm)\\
&+2\int_u^\infty s\left(\frac{s}{\sqrt{s^2-u^2}}-1\right)\frac{I_0(as)I_0(bs)}{\mathrm{e}^{\pi s}+1}\,\mathrm{d}s\\
&+2\int_0^u s\left(\frac{\mathrm{i}s}{\sqrt{u^2-s^2}}-1\right)\frac{I_0(as)I_0(bs)}{1+\mathrm{e}^{\pi s}}\,\mathrm{d}s=0,
\end{aligned}
\tag{3}
$$

where $I_0(x)$ is the modified Bessel function of the first kind of order zero.

Taking real part only in (3), the infinite series (1) is now expressed in terms of infinite integrals as

$$
\begin{aligned}
&\sum_{m=1,3}^{\infty} m\left(\frac{m}{\sqrt{m^2+u^2}}-1\right)J_0(am)J_0(bm)\\
&=\tfrac{1}{2}\int_0^\infty t\left(\frac{t}{\sqrt{t^2+u^2}}-1\right)J_0(at)J_0(bt)\,\mathrm{d}t\\
&+\int_u^\infty \frac{s^2}{\sqrt{s^2-u^2}}\frac{I_0(as)I_0(bs)}{\mathrm{e}^{\pi s}+1}\,\mathrm{d}s-\int_0^\infty \frac{sI_0(as)I_0(bs)}{\mathrm{e}^{\pi s}+1}\,\mathrm{d}s,\quad \text{for } a+b<\pi.
\end{aligned}
\tag{4}
$$

In (4) the last two infinite integrals do not give trouble for computations. The first one is difficult to calculate due to the oscillatory behaviour of the Bessel function $J_0(x)$. Therefore, we need to transform that infinite integral to a much more computable form.

3. Evaluation of the infinite integral

We used the identity introduced by Eason et al. [2] which is

$$J_0(at)J_0(bt)=\frac{1}{\pi}\int_0^\pi J_0\left(t(a^2+b^2-2ab\cos\theta)^{1/2}\right)\mathrm{d}\theta,$$

for the product of Bessel functions and so the first infinite integral in (4) takes the form

$$\tfrac{1}{2}\int_0^\infty t\left(\frac{t}{\sqrt{t^2+u^2}}-1\right)J_0(at)J_0(bt)\,\mathrm{d}t=\frac{1}{2\pi}\int_0^\infty\int_0^\pi t\left(\frac{t}{\sqrt{t^2+u^2}}-1\right)J_0(\gamma t)\,\mathrm{d}\theta\,\mathrm{d}t,\tag{5}$$

where $\gamma^2=a^2+b^2-2ab\cos\theta$.

Now, we consider the infinite integral (change the order of integrations) in the double integral (5)

$$\int_0^\infty t\left(\frac{t}{\sqrt{t^2+u^2}}-1\right)J_0(\gamma t)\,\mathrm{d}t,$$

and try to compute it in terms of modified Bessel functions. So, put $t = u \text{ sh } v$ in the above integral to obtain

$$\int_0^\infty t\left(\frac{t}{\sqrt{t^2+u^2}} - 1\right) J_0(\gamma t)\, \mathrm{d}t = u^2 \int_0^\infty (\text{sh } v - \text{ch } v) J_0(\gamma u \text{ sh } v) \text{sh } v\, \mathrm{d}v,$$

where sh x and ch x are sine hyperbolic and cosine hyperbolic functions respectively.

From the identity

$$J_1(z) + z\frac{\mathrm{d}}{\mathrm{d}z} J_1(z) = zJ_0(z),$$

where $J_1(z)$ is the first kind Bessel function of order one, we take $J_0(\gamma u \text{ sh } v)$ and substitute it in the above integral to get

$$\int_0^\infty t\left(\frac{t}{\sqrt{t^2+u^2}} - 1\right) J_0(\gamma t)\, \mathrm{d}t$$
$$= u^2 \int_0^\infty (\text{sh } v - \text{ch } v)\left[\frac{1}{\gamma u}(\gamma u \text{ sh } v) + \frac{1}{\gamma}\frac{\mathrm{d}}{\mathrm{d}u} J_1(\gamma u \text{ sh } v)\right] \mathrm{d}v. \tag{6}$$

By using the identities [3, Section 6.663] for $\mu = 1$, $\nu = 0$ and $z > 0$:

$$\int_0^\infty J_1(2z \text{ sh } t)\text{ch } t\, \mathrm{d}t = \tfrac{1}{2}[I_0(z)K_1(z) + I_1(z)K_0(z)],$$
$$\int_0^\infty J_1(2z \text{ sh } t)\text{sh } t\, \mathrm{d}t = \tfrac{1}{2}[I_0(z)K_1(z) - I_1(z)K_0(z)], \tag{7}$$

one can easily obtain the infinite integral as

$$\int_0^\infty t\left(\frac{t}{\sqrt{t^2+u^2}} - 1\right) J_0(\gamma t)\, \mathrm{d}t$$
$$= -\frac{u}{\gamma} I_1\left(\frac{\gamma u}{2}\right) K_0\left(\frac{\gamma u}{2}\right) - \frac{u^2}{\gamma}\left[\frac{\mathrm{d}}{\mathrm{d}u} I_1\left(\frac{\gamma u}{2}\right) K_0\left(\frac{\gamma u}{2}\right) + I_1\left(\frac{\gamma u}{2}\right)\frac{\mathrm{d}}{\mathrm{d}u} K_0\left(\frac{\gamma u}{2}\right)\right], \tag{8}$$

where $I_0(x)$ is the modified Bessel function of the first kind of order one and $K_0(x)$, $K_1(x)$ are the modified Bessel functions of the second kind of order zero and one, respectively.

Finally, using the identities

$$I_1(z) + z\frac{\mathrm{d}}{\mathrm{d}z} I_1(z) = zI_0(z), \qquad K_0'(z) = -K_1(z),$$

we obtain

$$\int_0^\infty t\left(\frac{t}{\sqrt{u^2+t^2}} - 1\right) J_0(\gamma t)\, \mathrm{d}t = \tfrac{1}{2}u^2[I_1(\tfrac{1}{2}u\gamma)K_1(\tfrac{1}{2}u\gamma) - I_0(\tfrac{1}{2}u\gamma)K_0(\tfrac{1}{2}u\gamma)]. \tag{9}$$

So, the infinite series A in equation (1) is transformed now through equations (9), (5) and (4) into the form

$$\begin{aligned} A &= \sum_{m=1,3}^{\infty} m\left(\frac{m}{\sqrt{m^2+u^2}} - 1\right) J_0(am) J_0(bm) \\ &= \frac{u^2}{4\pi}\int_0^{\pi}\left[I_1\left(\frac{u\gamma}{2}\right)K_1\left(\frac{u\gamma}{2}\right) - I_0\left(\frac{u\gamma}{2}\right)K_0\left(\frac{u\gamma}{2}\right)\right] \mathrm{d}\theta \\ &\quad + \int_u^{\infty} \frac{s^2}{\sqrt{s^2-u^2}} \frac{I_0(as) I_0(bs)}{\mathrm{e}^{\pi s}+1} \mathrm{d}s - \int_0^{\infty} \frac{s I_0(as) I_0(bs)}{\mathrm{e}^{\pi s}+1} \mathrm{d}s, \quad \text{for } a+b<\pi. \end{aligned} \tag{10}$$

As one can notice, the infinite series (1) now is expressed in terms of a finite integral containing modified Bessel functions only and two infinite integrals giving no computational difficulties. This form of the infinite series (1) is calculable and the numerical values are obtained for several values of the constants u, a and b.

4. Numerical results and discussion

For computing the value of the infinite series A for several values of a, b and u, we need to compute I_1, K_1, I_0, K_0 (the modified Bessel functions). These functions were evaluated at the required arguments by using a subprogram from SSP package in single precision on Burroughs machine at the Middle East Technical University. In the transformed form of series A (10) the first integral is a finite integral giving no computational trouble except for $a = b$ with small values of θ. Since $K_0(x)$ and $K_1(x)$ behave like $\ln x$ and $1/x$, respectively, for small arguments, we make the $\theta = \pi x^2$ substitution in that integral to deal with the singularity at $a = b$. Then the trapezoidal rule was used with $n = 100$.

The second infinite integral from u to ∞ is negligible for large u and difficult to compute for small values of u. This integral is transformed to the finite form (by taking first $y = u/s$, and then with $y = \sin\theta$)

$$u^2\int_0^{\frac{\pi}{2}} \frac{1}{\sin^3\theta} \frac{\mathrm{e}^{-\pi u/\sin\theta} I_0\left(\frac{au}{\sin\theta}\right) I_0\left(\frac{bu}{\sin\theta}\right)}{1+\mathrm{e}^{-\pi u/\sin\theta}} \mathrm{d}\theta.$$

This way, the difficulties with small θ are handled.

The third infinite integral is also transformed into a finite integral of the form

$$\int_0^1\left[\frac{s\, I_0(as) I_0(bs)\, \mathrm{e}^{-\pi s}}{1+\mathrm{e}^{-\pi s}} + \frac{1}{s^3} \frac{\mathrm{e}^{-\pi s} I_0\left(\frac{a}{s}\right) I_0\left(\frac{b}{s}\right)}{1+\mathrm{e}^{-\pi/s}}\right] \mathrm{d}s.$$

For all these finite integrals the trapezoidal rule with $n = 100$ was used for several values of a, b and u.

There was no difficulty now for computations with this transformed form of the series A.

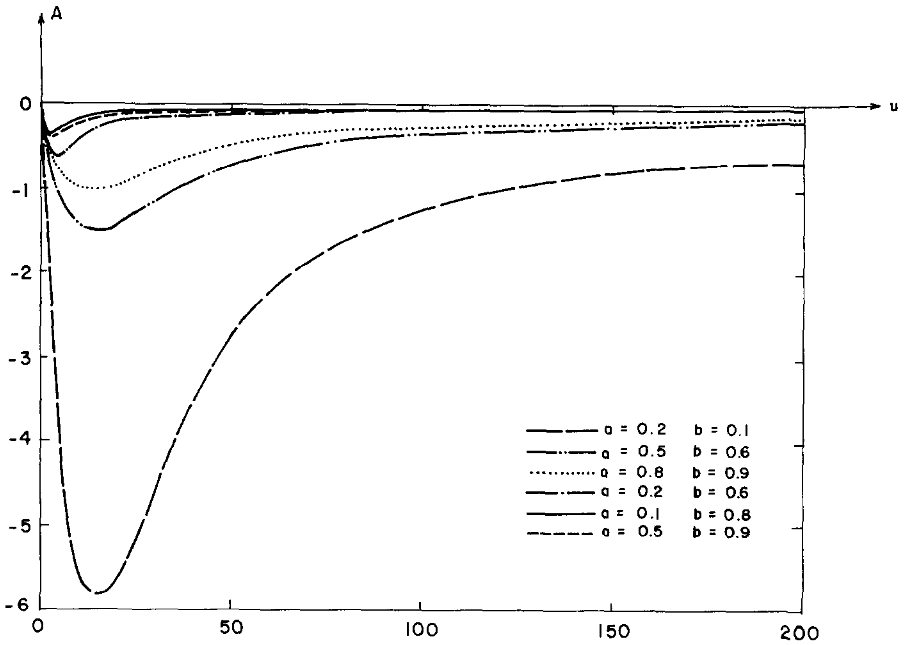

Fig. 2.

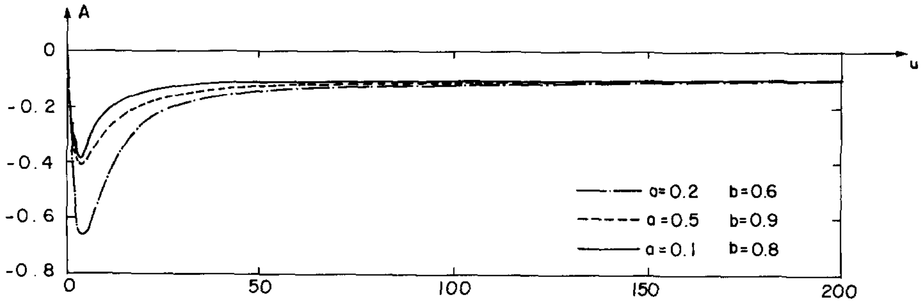

Fig. 3.

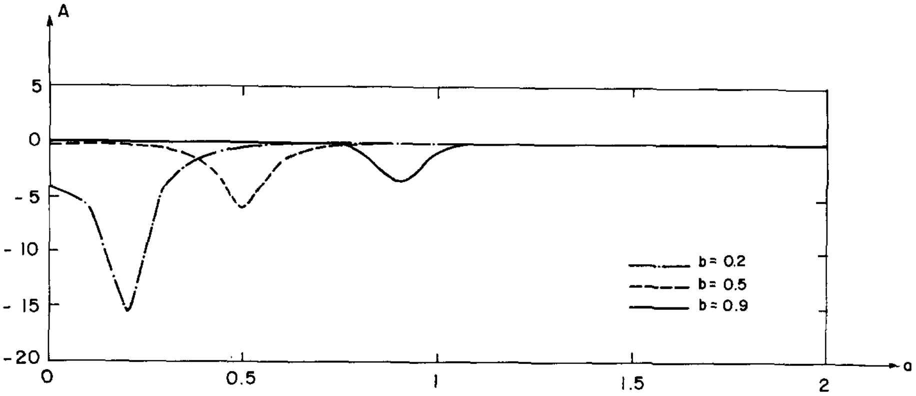

Fig. 4. $u = 20$.

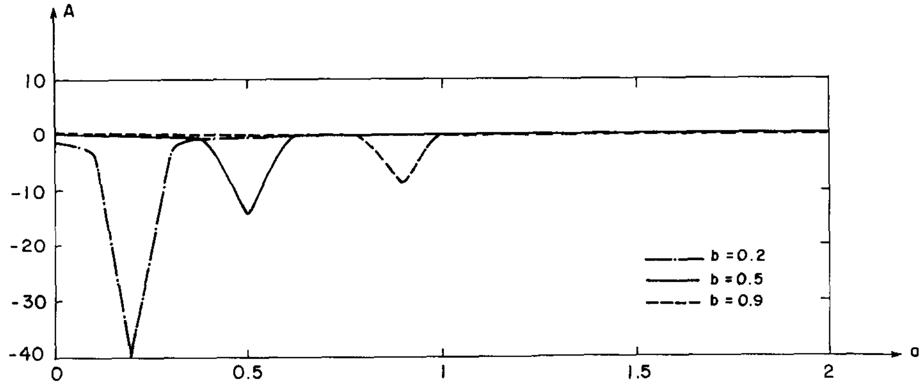

Fig. 5. $u = 50$.

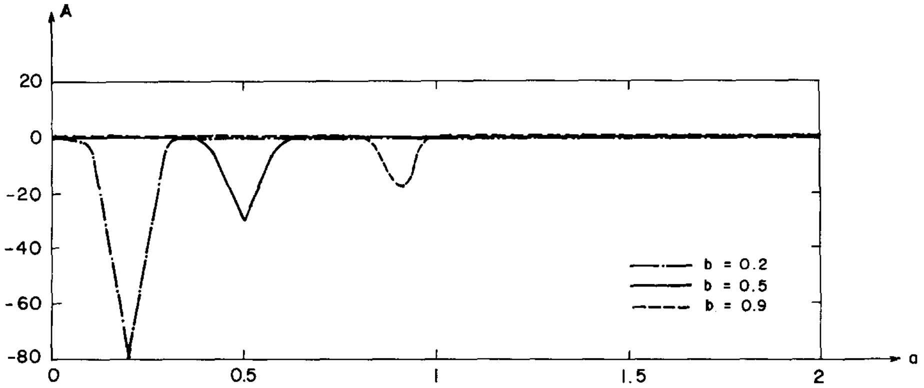

Fig. 6. $u = 100$.

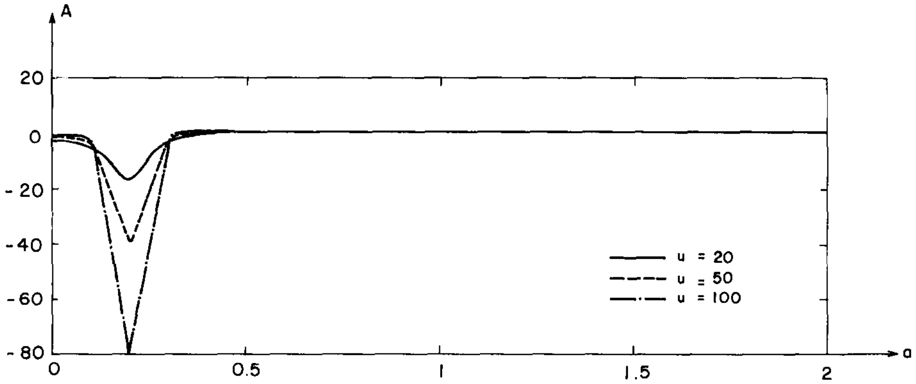

Fig. 7. $b = 0.2$.

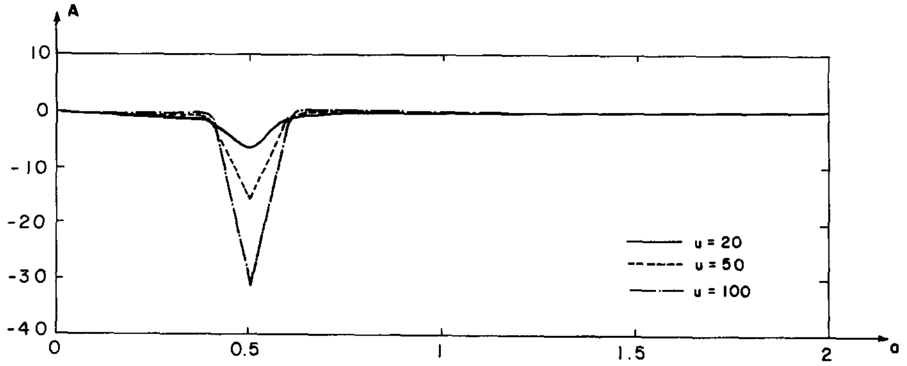

Fig. 8. $b = 0.5$.

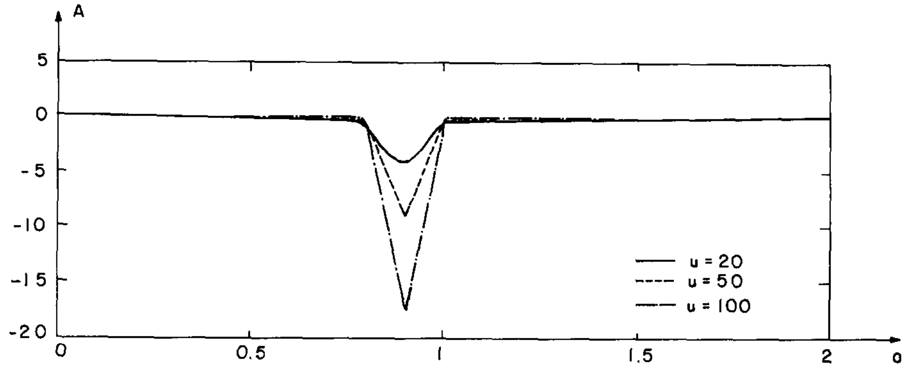

Fig. 9. $b = 0.9$.

Figures 2–9 show the behaviour of infinite series A for several values of a, b and u. The ranges of the variables a and b were kept less than two since those variables correspond to the dimensions of ducts, cracks, etc., in the physical problems while the variable u can go up to 200.

References

[1] L. Dragoş, *Magnetofluid Dynamics* (Abacus Press, Tunbridge Wells, 1975).

[2] G. Eason, B. Nobel and I.N. Sneddon, On certain integrals of Lipschitz–Hankel type involving products of Bessel functions, *Philos. Trans. Roy. Soc. London Ser. A* **247** (1955) 529–551.

[3] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals Series and Products* (Academic Press, New York, 1965).

[4] I.N. Sneddon, *Mixed Boundary Value Problems in Potential Theory* (North-Holland, New York, 1966).